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Transitive Novikov Algebras on Four-Dimensional Nilpotent Lie Algebras

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Novikov algebras were introduced in connection with the Poisson brackets (of hydrodynamic type) and Hamiltonian operators in the formal variational calculus. The commutator of a Novikov algebra is a Lie algebra, and the radical of a finite-dimensional Novikov algebra is transitive. In this paper, we give a classification of transitive Novikov algebras on four-dimensional nilpotent Lie algebras based on Kim (1986, *Journal of Differential Geometry* **24**, 373–394).

1. INTRODUCTION

One remarkable feature of Hamiltonian operators is their connection with certain algebraic structures (Balinskii and Novikov, 1985; Dubrovin and Novikov, 1983, 1984; Gel'fand and Diki, 1975, 1976; Gel'fand and Dorfman, 1979; Xu, 1995 a,b). Gel'fand and Diki introduced formal variational calculus and found certain interesting Poisson structures when they studied Hamiltonian systems related to certain nonlinear partial differential equations, such as KdV equations (Gel'fand and Diki, 1975, 1976). Gel'fand and Dorfman (1979) found more connections between Hamiltonian operators and certain algebraic structures. Dubrovin, Balanskii, and Novikov studied similar Poisson structures from another point of view (Balinskii and Novikov, 1985; Dubrovin and Novikov, 1983, 1984). One of the algebraic structures appearing in Gel'fand and Dorfman (1979) and Balinskii and Novikov (1985), which is called a "Novikov algebra" by Osborn (Osborn, 1992a,b,

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1994; Xu, 1996, 1997, 2000), was introduced in connection with the Poisson brackets of the hydrodynamic type.

A Novikov algebra A is a vector space over a field $\mathbf{K}$ with a bilinear product $(x, y) \to xy$ satisfying

$$(x_1, x_2, x_3) = (x_2, x_1, x_3) \tag{1.1}$$

and

$$(x_1x_2)x_3 = (x_1x_3)x_2, \tag{1.2}$$

for $x_1, x_2, x_3 \in A$, where

$$(x_1, x_2, x_3) = (x_1x_2)x_3 - x_1(x_2x_3). \tag{1.3}$$

Novikov algebras are a special class of left-symmetric algebras that only satisfy Eq. (1.1). Left-symmetric algebras are nonassociative algebras arising from the study of affine manifolds, affine structures and convex homogeneous cones (Bai and Meng, 2000; Burde, 1998; Kim, 1986; Vinberg, 1963).

The commutator of a Novikov algebra (or a left-symmetric algebra) A

$$[x, y] = xy - yx, \tag{1.4}$$

defines a (subadjacent) Lie algebra $\mathcal{G} = \mathcal{G}(A)$. Let L_x and R_x denote the left and right multiplication respectively, i.e., $L_x(y) = xy$, $R_x(y) = yx$, $\forall x, y \in A$; then for a Novikov algebra, the left multiplication operators form a Lie algebra and the right multiplication operators are commutative.

Zel'manov (1987) gave a fundamental structure theory of a finite-dimensional Novikov algebra over an algebraically closed field with characteristic 0: A Novikov algebra A is called right-nilpotent or transitive if every R_x is nipotent. Then by Eq. (1.2), a finite-dimensional Novikov algebra A contains the (unique) largest transitive ideal $N(A)$ (called the radical of A), and the quotient algebra $A/N(A)$ is a direct sum of fields. The transitivity corresponds to the completeness of the affine manifolds in geometry (Kim, 1986; Vinberg, 1963). It is well-known that the sub-adjacent Lie algebra of a transitive left-symmetric algebra is solvable (Kim, 1986).

Therefore, for further understanding and application, it is necessary to take the study of transitive Novikov algebras as the first step. Obviously, it is quite difficult to give a detailed structure theory for the transitive Novikov algebras because of the nonassociativity. We have obtained the classification of

Novikov algebras in dimensions 2 and 3 in Bai and Meng (2001). However, the method used there is useless for the cases in high dimensions, even in dimension 4.

On the other hand, for reasons of geometry, the study of transitive left-symmetric algebras on nilpotent Lie algebras is also to be taken as the first step (Kim, 1986), which will be used to construct a general theory on all solvable Lie algebras. Moreover, Kim (1986), using an extension theory, has given a classification of transitive left-symmetric algebras on four-dimensional nilpotent Lie algebras over the real-number field **R**.

Hence we give a classification, based on the classification by Kim (1986), of transitive Novikov algebras on four-dimensional nilpotent Lie algebras in this paper. We hope that the study of these cases can serve as a guide for further development. We would like point out that the method in Kim (1986) is less useful for the study in higher dimensions or for the cases on non-nilpotent Lie algebras. It is still an open question on the complete classification of transitive Novikov algebras in dimension ≥ 4.

Let $\{e_1, e_2, e_3, e_4\}$ be a basis. As is well-known, there are three four-dimensional nilpotent Lie algebras up to isomorphism:

$$\begin{aligned} A &= \langle e_1,\ e_2,\ e_3,\ e_4 \mid [e_i,\ e_j] = 0\rangle \text{ abelian} \\ H &= \langle e_1,\ e_2,\ e_3,\ e_4 \mid [e_2,\ e_3] = e_1, \text{ other products are zero}\rangle \\ T &= \langle e_1,\ e_2,\ e_3,\ e_4 \mid [e_2,\ e_3] = e_1,\ [e_3,\ e_4] = e_2, \text{ other products are zero}\rangle \end{aligned}$$

In the following sections, we give our classifications according to the above three cases receptively.

2. THE ABELIAN LIE ALGEBRAS (A)

It is easy to show that a transitive Novikov algebra on an abelian Lie algebra must be a commutative nilpotent associative algebra.

Recall that the (form) characteristic matrix of a Novikov algebra is defined as

$$\mathcal{A} = \begin{pmatrix} \sum_{k=1}^{n} a_{11}^k e_k & \cdots & \sum_{k=1}^{n} a_{1n}^k e_k \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^{n} a_{n1}^k e_k & \cdots & \sum_{k=1}^{n} a_{nn}^k e_k \end{pmatrix}$$

where $\{e_i\}$ is a basis of A and $e_i e_j = \sum_{k=1}^{n} a_{ij}^k e_k$. By Kim (1986) and using the condition $R_{e_i}R_{e_j} = R_{e_j}R_{e_i}$, the classification of transitive Novikov algebras on (A) can be given as in the following table:

Characteristic matrix	Symbols in Kim (1986)	Characteristic matrix	Symbols in Kim (1986)
(A1) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 \\ 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & e_1 \end{pmatrix}$	(3)	(A2) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 \\ 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & -e_1 \end{pmatrix}$	(4)
(A3) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & e_1 & 0 \\ 0 & e_1 & 0 & e_2 \end{pmatrix}$	$(30)_1$	(A4) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & e_1 & e_2 \\ 0 & e_1 & e_2 & e_3 \end{pmatrix}$	$(41)_{1,1}$
(A5) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & e_1 \end{pmatrix}$	$(51)_0$	(A6) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & -e_1 \end{pmatrix}$	$(52)_0$
(A7) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_2 \\ 0 & 0 & e_2 & -e_1 \end{pmatrix}$	$(53)_0$	(A8) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & e_2 \end{pmatrix}$	(54)
(A9) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_2 \\ 0 & 0 & e_2 & 0 \end{pmatrix}$	$(57)_0$	(A10) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & e_1 & e_3 \end{pmatrix}$	$(60)_1$
(A11) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \end{pmatrix}$	(61)	(A12) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	(62)

3. THE LIE ALGEBRA (H)

This Lie algebra is isomorphic to a direct sum of a Heisenberg Lie algebra and the real number **R**. From Kim (1986) and by using the condition $R_{e_i}R_{e_j} = R_{e_j}R_{e_i}$, we can give the classification of transitive Novikov algebras on (H) in the following table:

Characteristic matrix	Associativity	Symbols in Kim (1986)	Characteristic matrix	Associativity	Symbols in Kim (1986)
(H1) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_1 & e_1 & 0 \\ 0 & -e_1 & 0 & e_1 \\ 0 & 0 & e_1 & 0 \end{pmatrix}$	Associative	(5)	(H2) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & 0 \\ 0 & -e_1 & 0 & e_1 \\ 0 & 0 & e_1 & 0 \end{pmatrix}$	Associative	(6)
(H3) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_1 & e_1 & 0 \\ 0 & -e_1 & te_1 & 0 \\ 0 & 0 & 0 & e_1 \end{pmatrix}$	Associative	$(7)_t$	(H4) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_1 & e_1 & 0 \\ 0 & 0 & te_1 & 0 \\ 0 & 0 & 0 & -e_1 \end{pmatrix}$ $t \geq 0$	Associative	$(8)_t$ $t \geq 0$
(H5) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & e_2 \end{pmatrix}$	Nonassociative	(28)	(H6) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_2 \end{pmatrix}$	Nonassociative	(29)
(H7) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & te_1 \\ 0 & 0 & e_1 & 0 \\ 0 & e_1 & 0 & e_2 \end{pmatrix}$ $t \neq 1$	Nonassociative	$(30)_t$ $t \neq 1$	(H8) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & te_1 \\ 0 & 0 & 0 & e_1 \\ 0 & e_1 & 0 & e_2 \end{pmatrix}$	Associative ($t = 1$) Nonassociative ($t \neq 1$)	$(31)_t$
(H9) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & e_1 & e_2 \\ 0 & e_1 & e_1 + e_2 & e_3 \end{pmatrix}$	Nonassociative	$(40)_1$	(H10) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_2 \\ 0 & 0 & -e_2 & 0 \end{pmatrix}$	Associative	(46)
(H11) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_1 \\ 0 & 0 & -e_1 & 0 \end{pmatrix}$	Associative	(47)	(H12) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_2 \\ 0 & 0 & -e_2 & 0 \end{pmatrix}$	Associative	(48)

(Continued)

	Characteristic matrix	Associativity	Symbols in Kim (1986)		Characteristic matrix	Associativity	Symbols in Kim (1986)
(H13)	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_2 \\ 0 & 0 & -e_2 & e_1 \end{pmatrix}$	Associative	(49)	(H14)	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_2 \\ 0 & 0 & -e_2 & -e_1 \end{pmatrix}$	Associative	(50)
(H15)	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & te_1 \\ 0 & 0 & -te_1 & e_1 \end{pmatrix}$ $t > 0$	Associative	$(51)_t$ $t > 0$	(H16)	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & te_1 \\ 0 & 0 & -te_1 & -e_1 \end{pmatrix}$ $t > 0$	Associative	$(52)_t$ $t > 0$
(H17)	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_3 \end{pmatrix}$	Associative	(58)	(H18)	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & e_2 & e_3 \end{pmatrix}$	Associative	(59)
(H19)	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & te_1 & e_3 \end{pmatrix}$ $t \neq 1$	Associative	$(60)_t$ $t \neq 1$	(H20)	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & (1+t)e_2 \\ 0 & 0 & (1-t)e_2 & -e_1 \end{pmatrix}$ $t > 0$	Associative	$(53)_t$ $t > 0$
(H21)	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_1 + te_2 \\ 0 & 0 & -e_1 - te_2 & e_2 \end{pmatrix}$	Associative	$(55)_t$	(H22)	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_1 + e_2 \\ 0 & 0 & -e_1 + e_2 & 0 \end{pmatrix}$	Associative	(56)
(H23)	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & (1+t)e_2 \\ 0 & 0 & (1-t)e_2 & 0 \end{pmatrix}$ $t > 0$	Associative	$(57)_t$ $t > 0$				

4. THE LIE ALGEBRA (T)

This Lie algebra is three-step nilpotent, that is, the dimension of its derivation $[A, A]$ is 2. From Kim (1986) and by using the condition $R_{e_i} R_{e_j} = R_{e_j} R_{e_i}$, we can give the classification of transitive Novikov algebras on (T) in the following table:

Characteristic matrix		Associativity	Symbols in Kim (1986)
(T1) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_1 \\ 0 & 0 & e_2 & e_2 \\ 0 & -2e_1 & -e_2 & -e_2 \end{pmatrix}$		Nonassociative	$(18)_{-1}$
(T2) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_1 \\ 0 & 0 & e_2 & e_2 \\ 0 & -2e_1 & e_1 - e_2 & -e_2 \end{pmatrix}$		Nonassociative	$(19)_{-1}$
(T3) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_1 \\ 0 & 0 & e_1 + e_2 & e_2 \\ 0 & -2e_1 & te_1 - e_2 & -e_2 \end{pmatrix}$		Nonassociative	$(20)_{-1,t}$
(T4) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_1 \\ 0 & 0 & 0 & e_2 \\ 0 & e_1 & 0 & e_3 \end{pmatrix}$		Nonassociative	(35)
(T5) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & te_1 & e_2 \\ 0 & (2t-1)e_1 & te_2 & e_3 \end{pmatrix}$	$t \neq 1$	Nonassociative	$(41)_{t,t}$ $t \neq 1$
(T6) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & e_1 & e_2 & e_3 \end{pmatrix}$		Nonassociative	(42)
(T7) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & e_1 & e_2 & e_3 \end{pmatrix}$		Nonassociative	(43)

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